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THEORETICAL LIFT DISTRIBUTION AND
UPWASH VELOCITIES FOR THIN WINGS
AT SUPERSONIC SPEEDS

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SUMMARY

A method for calculating the upwash-velocity component in the vicinity of thin wings at supersonic speeds is presented. The method is applied to obtain an explicit expression for the upwash over wing tips of fairly general plan form and profile. As a special case, numerical values are presented for the slopes of the streamlines off the tip of a rectangular plan-form thin flat-plate wing. The formulation is extended to give a method for obtaining the velocity potential at points on arbitrary thin wings influenced by isolated or interacting external flow fields off the wing plan form. The solutions obtained by the method for regions influenced by so-called subsonic trailing edges do not conform, however, to the Kutta-Joukowski condition.

The method in principle may be applied to obtain the aerodynamic coefficients and hence the lift distribution for thin wings of arbitrary plan form and profile; the calculus involved in obtaining explicit solutions, however, is likely to be difficult and impractical. The functions were therefore altered to isolate and to remove nonessential singularities. The equations so obtained are suitable for numerical calculations of the aerodynamic coefficients of arbitrary thin wings at supersonic speeds. As an example of the method, the upwash between the leading edges and the foremost Mach waves in the plane of the flat-plate delta wing was calculated and compared with approximate results obtained by neglecting flow-field interaction. The pressure coefficient on the surface of the wing was likewise numerically computed and compared with the exact solution.

INTRODUCTION

A method for obtaining the lift, drag, and pressure distributions in the vicinity of thin wing tips at supersonic speeds is presented in reference 1. The basis of the method was to place a thin diaphragm along a stream sheet in the plane of the wing between the wing boundary and the foremost Mach wave. In this manner an integral equation was established to define the slopes of the streamlines in the disturbed flow field. The interaction effects of the two surfaces of the wing in the vicinity of the tip were thus evaluated.

The main object of reference 1 was to present a method for calculating the theoretical aerodynamic coefficients on the surface of a wing. These coefficients were obtained without an explicit solution for the slope of the stream sheet for points on the wing surface influenced by noninteracting external flow fields. If the slopes of the streamlines could be determined, the upwash-velocity components in the external flow fields and the aerodynamic effects of interacting external flow fields could be evaluated. Pressure distributions and the lift and drag coefficients could then be calculated for arbitrary thin wings at supersonic speeds.

The present report shows that the defining equation for the slopes of the streamlines off the wing-plan-form boundaries is a special case of Abel's integral equation. The solution to Abel's equation is applied in order to obtain the slopes of the streamlines off wings of fairly general plan form and profile. As a special case, numerical values are presented for the slopes of the streamlines off the wing tip of a rectangular plan-form thin flat plate. The general formulation is shown to give a method for obtaining the velocity potential of finite wings influenced by interacting external flow fields. The solutions obtained by the method for regions influenced by so-called subsonic trailing edges do not conform, however, to the Kutta-Joukowski condition,

Although the method in principle may be applied to obtain the aerodynamic coefficients and hence the lift distribution for thin wings of arbitrary plan form and profiles, the calculus involved in obtaining explicit solutions is likely to be difficult and impractical. The functions are therefore altered to isolate and remove nonessential singularities. The equations so obtained are suitable for numerical calculations of the aerodynamic coefficients of arbitrary thin wings at supersonic speeds. As an example of the method, the pressure coefficient of the delta wing included within the Mach cone is computed by numerical methods and compared with the solution obtained by other methods.

SLOPES OF THE STREAMLINES

The analysis is considerably simplified in a set of oblique coordinates u, v whose axes lie parallel to the Mach lines of the flow. In this set, the value of one of the coordinates of a point is the distance measured parallel to the coordinate axis from the point to the other coordinate axis. If the wing lies in the x, y plane and the free-stream velocity is parallel to the x axis, the transformation equations from one set of coordinates to the other are

$$\begin{aligned}
 u &= \frac{M}{2\beta} (x - \beta y) & v &= \frac{M}{2\beta} (x + \beta y) \\
 x &= \frac{\beta}{M} (u + v) & y &= \frac{1}{M} (v - u)
 \end{aligned} \tag{1}$$

where M is the Mach number and $\beta \equiv \sqrt{M^2 - 1}$. (A list of symbols is included in appendix A.)

If the coordinate zero of the two systems is placed on the point of tangency of the wing boundary (fig. 1) and the foremost Mach wave and if the wing boundary is represented by the two equations $v = v_1(u)$ and $v = v_2(u)$, the defining equation for the slope λ of the stream sheet (referred to as a diaphragm) near the plane of the wing measured in $y = \text{constant}$ planes is given in reference 1 as

$$\int_0^{u_D} \frac{du}{\sqrt{u_D - u}} \int_{v_2(u)}^{v_D} \frac{\lambda(u, v) dv}{\sqrt{v_D - v}} = \int_0^{u_D} \frac{du}{\sqrt{u_D - u}} \int_{v_1(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2\sqrt{v_D - v}} \tag{2}$$

where u_D and v_D represent the coordinates of a local point on the diaphragm and σ_B and σ_T are the slopes (measured in radians) of the wing on the bottom and top surfaces, respectively. (The sign of the wing slope is defined oppositely on the bottom and top surfaces; for example, σ_B and σ_T are both positive for a wedge-profile wing at zero angle of attack.)

The derivation of equation (2) presented in reference 1 required that the three components of the perturbation velocity be continuous across the diaphragm. If the wing sheds a vortex sheet, discontinuities in one or more of the perturbation-velocity components are feasible. Because the diaphragm can sustain no pressure difference between its top and bottom surfaces, the x component of the perturbation velocity must be continuous. The partial derivative with respect to x of equation (2) then applies rather than equation (2). Integration yields equation (2), except that an arbitrary function of y may be added to either member. This function of y represents the circulation in the vortex sheet and may be adjusted to make the solution to the equation satisfy the Kutta-Joukowski condition in cases involving so-called subsonic trailing edges.

Inasmuch as the limits of integration with respect to u are the same for both members of equation (2) for all values of u_D and because of the nature of the functions, the equation may be reduced, as in reference 1, to the form

$$\int_{v_2(u)}^{v_D} \frac{\lambda(u, v) dv}{\sqrt{v_D - v}} = \int_{v_1(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2\sqrt{v_D - v}} \quad (3)$$

Equation (3) is a special case of Abel's integral equation. In the notation of reference 2, if

$$\int_{\alpha}^x \frac{u(\xi)}{(x - \xi)^{\mu}} d\xi = f(x) \quad \begin{matrix} 0 < \mu < 1 \\ \alpha \leq x \leq b \end{matrix} \quad (4)$$

where x, α , and b are real and finite, then "the continuous solution, if it exists, can be none other than"

$$u(z) = \frac{\sin \mu\pi}{\pi} \frac{d}{dz} \int_{\alpha}^z \frac{f(x) dx}{(z - x)^{1-\mu}} \quad (5)$$

The following table compares the notation of the symbols of equations (3) and (4):

Notation of	
Equation (4)	Equation (3)
$f(x)$	$f(v_D) = \int_{v_1(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2\sqrt{v_D - v}}$
x	v_D
α	$v_2(u)$
ξ	v
$u(\xi)$	$\lambda(u, v)$
μ	$\frac{1}{2}$

(The function $f(v_D)$ is usually not zero at $v_D = v_2(u)$ nor is $f'(v_D)$ continuous, as required in reference 2. A singularity exists in the present problem that may be isolated. For a discussion of the requirements for application of equation (5) to obtain a solution of equation (4), see reference 3.) The solution of equation (3) is then given by equation (5) as

$$\lambda(u, z) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \int_{v_1(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2\sqrt{v_D-v}} \quad (6)$$

evaluated at $z = v$.

The upwash-velocity component V_z is directly related to λ by the equation

$$V_z = \lambda U. \quad (7)$$

where U is the free-stream velocity. Inasmuch as the sign of λ was chosen with respect to the top surface of the diaphragm, a positive λ implies an upflow. Substitution of λ from equation (6) into equation (7) yields

$$V_z = \frac{U}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \int_{v_1(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2\sqrt{v_D-v}} \quad (8)$$

evaluated at $z = v$.

Equation (6) is relatively easy to evaluate when the profile of the wing is symmetrical about the plane of the wing. In this case, $(\sigma_B - \sigma_T) = 2\alpha$ where α is the angle of attack. The indicated integration for the wing plan form of figure 1 is carried out in appendix B as equation (B5) to give

$$\lambda(u, v) = \frac{2\alpha}{\pi} \left[\sqrt{\frac{v_2(u) - v_1(u)}{v - v_2(u)}} - \tan^{-1} \sqrt{\frac{v_2(u) - v_1(u)}{v - v_2(u)}} \right] \quad (9)$$

That equation (9) is the solution of equation (3) for the symmetrical wing has been verified by direct substitution. (See appendix B.) (If $\sigma_B - \sigma_T$ is a function only of u , the solution, equation 9, for λ still applies providing the factor 2α is replaced by $\sigma_B - \sigma_T$.)

A plot of λ/α (from equation (9)) as a function of the independent variable $\frac{v_2(u) - v_1(u)}{v - v_2(u)}$ is presented in figure 2. Along the foremost Mach wave, $v_2(u) - v_1(u) = 0$ and $\lambda = 0$. The value of λ becomes infinite at the wing boundary corresponding to the mathematical discontinuity along $v = v_2(u)$.

The quantity $\frac{v_2(u) - v_1(u)}{v - v_2(u)}$ may be interpreted geometrically with the aid of figure 1. The quantity $v_2(u) - v_1(u)$ is the length of the line $a b$, whereas $v - v_2(u)$ is the length of the line $b v$. From figure 2 it is then apparent that v must be close to the wing boundary before λ/α becomes very large.

As a further illustration, the diaphragm slope of a rectangular plan-form thin flat-plate wing has been calculated. In this case, $v_1(u) = -u$, $v_2(u) = u$, and by equations (1)

$$\frac{v_2(u) - v_1(u)}{v - v_2(u)} = \frac{2u}{v - u} = \frac{x}{\beta y} - 1 \quad (10)$$

The quantity $\beta y/x$ was therefore taken as the independent variable. The resulting slope ratio λ/α is presented in figure 3 as a function of $\beta y/x$. The ratio λ/α remains fairly small except near the wing edge. For example, at a Mach number of 3, λ/α is still only 4.26 for $y/x = 0.0053$.

APPLICATIONS TO AERODYNAMIC THEORY

The explicit solution (equation (6)) of the integral equation (3) has direct application in the calculation of the lift distributions of thin wings. The solutions obtained for cases involving so-called subsonic trailing edges may violate the Kutta-Joukowski condition and hence should be used with caution. In reference 1 a method is presented for obtaining the aerodynamic coefficients in the vicinity of wing tips under the influence of independently disturbed external flow fields; that is, each disturbed external flow field includes no other external unknown flow

field in its forward Mach cone. By application of equation (6), portions of the external flow field may be evaluated. The interaction effects of one external field on another may be determined.

The details of the process will be illustrated for the wing plan form of figure 4. The leading edges are defined by the equations

$$v = v_1(u) \quad \text{or} \quad u = u_1(v)$$

$$v = v_2(u) \quad \text{or} \quad u = u_2(v)$$

$$v = v_3(u) \quad \text{or} \quad u = u_3(v)$$

The diaphragm area S_D may be divided into sections $S_{D,1}$ and $S_{D,2}$, where there is no interaction, and $S_{D,3}$ and $S_{D,4}$, where there is interaction. The slope λ of the diaphragm $S_{D,2}$ is given by equation (8) as

$$\lambda_2(u, z) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \int_{v_1(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2 \sqrt{v_D - v}} \quad (11)$$

evaluated at $z = v$. In a similar manner, the slope of the diaphragm $S_{D,1}$ is

$$\lambda_1(z, v) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{u_3(v)}^z \frac{du_D}{\sqrt{z-u_D}} \int_{u_1(v)}^{u_3(v)} \frac{(\sigma_B - \sigma_T) du}{2 \sqrt{u_D - u}} \quad (12)$$

evaluated at $z = u$.

Either of two schemes may now be applied to extend the calculation of the velocity potential beyond the shaded region. In calculating the influence of the external field $S_{D,2}$ and $S_{D,4}$, the diaphragm $S_{D,1}$ is considered as part of the wing; similarly, when the influence of the external fields $S_{D,1}$ and $S_{D,3}$ is to be calculated, the diaphragm $S_{D,2}$ is considered as part of the wing. The methods of reference 1 then directly apply.

The velocity potential ϕ_T at any point u_w, v_w on the top surface of the wing is given by the integral

$$\phi_T = -\frac{U}{\pi M} \iint_{S_W} \frac{\sigma_T du dv}{\sqrt{(u_w-u)(v_w-v)}} - \frac{U}{\pi M} \iint_{S_D} \frac{\lambda du dv}{\sqrt{(u_w-u)(v_w-v)}} \quad (13)$$

or in Cartesian coordinates

$$\phi_T = -\frac{U}{\pi} \iint_{S_W} \frac{\sigma_T d\xi d\eta}{\sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2}} - \frac{U}{\pi} \iint_{S_D} \frac{\lambda d\xi d\eta}{\sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2}} \quad (14)$$

where U is the free-stream velocity (parallel to the x axis) and S_W and S_D are the wing and diaphragm areas included in the forward Mach cone from the point u_w, v_w or x, y .

If for simplicity the quantity K is defined as

$$K(x, y, \xi, \eta) = \frac{1}{\pi \sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2}}$$

then in the notation of figure 5

$$\phi_T = - \iint_{S_W(1+2+3+4)} \sigma_T K d\xi d\eta - \iint_{S_D(1+2+3+4)} \lambda K d\xi d\eta \quad (15)$$

where the notation $S_W(1+2+3+4)$ means integration over the surface $S_{W,1} + S_{W,2} + \dots$. According to the methods of reference 1

$$\iint_{S_D(1+2+4)} \lambda K d\xi d\eta = \iint_{S_W(1+3)} \frac{(\sigma_B - \sigma_T)}{2} K d\xi d\eta \quad (16)$$

and

$$\iint_{S_D(1+2+3)} \lambda K d\xi d\eta = \iint_{S_W(1+2)} \frac{(\sigma_B - \sigma_T)}{2} K d\xi d\eta \quad (17)$$

Addition of equations (16) and (17) yields

$$\begin{aligned} \iint_{S_D(1+2+3+4)} \lambda K d\xi d\eta &= \iint_{S_W(2+3)} \frac{(\sigma_B - \sigma_T)}{2} K d\xi d\eta + \iint_{S_W,1} (\sigma_B - \sigma_T) K d\xi d\eta \\ &\quad - \iint_{S_D(1+2)} \lambda K d\xi d\eta \end{aligned} \quad (18)$$

Substitution of equation (18) into equation (15) yields

$$\begin{aligned} \varphi_T &= - \iint_{S_W,1} \sigma_B K d\xi d\eta - \iint_{S_W(2+3)} \frac{(\sigma_B + \sigma_T)}{2} K d\xi d\eta \\ &\quad - \iint_{S_W,4} \sigma_T K d\xi d\eta + \iint_{S_D,1} \lambda_1 K d\xi d\eta + \iint_{S_D,2} \lambda_2 K d\xi d\eta \end{aligned} \quad (19)$$

where λ_1 and λ_2 are given by equations (12) and (11), respectively.

An alternate scheme would be to calculate the slope of the diaphragm in regions $S_{D,3}$ and $S_{D,4}$. By applying equation (6),

$$\lambda_4(u, z) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \left[\int_0^{v_3(u)} \frac{-\lambda_1(u, v) dv}{\sqrt{v_D-v}} + \int_{v_3(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2\sqrt{v_D-v}} \right] \quad (20)$$

evaluated at $z = v$. Similarly

$$\lambda_3(z, v) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{u_3(v)}^z \frac{du_D}{\sqrt{z-u_D}} \left[\int_0^{u_2(v)} \frac{-\lambda_2(u, v) du}{\sqrt{u_D-u}} + \int_{u_2(v)}^{u_3(v)} \frac{(\sigma_B - \sigma_T) du}{2\sqrt{u_D-u}} \right] \quad (21)$$

evaluated at $z = u$. This process may be continued until the slopes of all the diaphragms are known. The velocity potential ϕ_T at any point (u_w, v_w) on the top surface of the wing is then given by equations (13) and (14).

One further case will be mentioned. If $v = v_1(u)$ reduces to a point at the origin, there is a continuous interaction of the two external fields. (See fig. 6.) The slope of the diaphragm in the region $S_{D,2}$ may be written as

$$\lambda_2(u, z) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \left[\int_0^{v_3(u)} \frac{-\lambda_1(u, v) dv}{\sqrt{v_D-v}} + \int_{v_3(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2\sqrt{v_D-v}} \right] \quad (22a)$$

evaluated at $z = v$ or

$$\lambda_2(u, z) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \int_{v_3(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) dv}{2\sqrt{v_D-v}} - \frac{1}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \int_0^{v_3(u)} \frac{\lambda_1(u, v) dv}{\sqrt{v_D-v}} \quad (22b)$$

evaluated at $z = v$. Similarly,

$$\lambda_1(z, v) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{u_3(v)}^z \frac{du_D}{\sqrt{z-u_D}} \int_{u_2(v)}^{u_3(v)} \frac{(\sigma_B - \sigma_T) du}{2\sqrt{u_D-u}} - \frac{1}{\pi} \frac{\partial}{\partial z} \int_{u_3(v)}^z \frac{du_D}{\sqrt{z-u_D}} \int_0^{u_2(v)} \frac{\lambda_2(u, v) du}{\sqrt{u_D-u}} \quad (23)$$

evaluated at $z = u$,

Thus there are two equations for the two unknown functions λ_1 and λ_2 . The functions λ_1 and λ_2 can therefore be determined, at least in principle, either by successive approximations or by direct substitution and solution of the resulting integral equation. (If the wing has a symmetrical plan form about the x axis, $\lambda_2(u, v) = \lambda_1(v, u)$ and the two equations (22) and (23) become a single integral equation. Furthermore, if the flow is conical, then λ_2 is a function of v/u .)

NUMERICAL CALCULATION OF DIAPHRAGM SLOPES

AND OF PERTURBATION VELOCITIES

Calculation of diaphragm slopes. - The calculation of the diaphragm slope λ and of the perturbation velocities $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$

in closed form or in series will generally be difficult for a wing of arbitrary form. Because the integrals defining λ and ϕ have finite regions of integration they are well suited to purely numerical integration.

In order to carry out numerical integration, it is always necessary to eliminate infinite singularities from the integrands. For the calculation of perturbation velocities, it is furthermore desirable to eliminate the extra step of the numerical differentiation of an integral.

The evaluation of the integrals involving singularities is discussed in appendix B. Equations are derived for the calculation of λ and of the derivatives of ϕ that are suitable for calculation with adding calculators or with the aid of mechanical computers.

The subsequent equations have been derived for a general wing with a mean camber line that is curved at the edges of the wing. If the curvature of the camber line is zero at the edges, the terms involving the derivatives of $(\sigma_B - \sigma_T)$ become zero. In the case of an uncambered wing, $(\sigma_B - \sigma_T)$ has the constant value of 2α . In these important cases the equations are simplified.

The final equations for $\lambda_2(u, v)$ equivalent to equations (20) to (22a) are derived in appendix C as equations (C13), (C16), (C24), and (C25). (See figs. 5 and 6.)

$$\lambda_2(u, v) = \frac{1}{\pi} \frac{g_2(u, v_2)}{\sqrt{v-v_2}} - \frac{1}{2} (\sigma_B - \sigma_T)_2 + \frac{1}{\pi} \int_{v_2}^v \frac{h'(v_D) - h'(\bar{v})}{\sqrt{v-v_D}} dv_D + \frac{2}{\pi} h'(v) \sqrt{v-v_2} \quad (24)$$

where

$$h^*(u, v_D) = \frac{(\sigma_B - \sigma_T)_2}{2\sqrt{v_D - v_3}} + \Lambda_1 \frac{\sqrt{v_3/v_d}}{v_D - v_3} + \left[\frac{(v_D - v_3) + (v_D - v_2)}{2\sqrt{v_D - v_3}} - \sqrt{v_D - v_2} \right] \left[\frac{d(\sigma_B - \sigma_T)}{dv} \right]_2$$

$$+ \frac{1}{2} \int_0^{v_3} \left(\lambda_1 - \frac{\Lambda_1}{\sqrt{v_3 - v}} \right) \frac{dv}{(v_D - v)^{3/2}} - \int_{v_3}^{v_2} \frac{(\sigma_B - \sigma_T) - (\sigma_B - \sigma_T)_2 - (v_D - v_2) \left[\frac{d(\sigma_B - \sigma_T)}{dv} \right]_2}{4(v_D - v)^{3/2}} dv \quad (25)$$

$$g_2(u, v_2) = (\sigma_B - \sigma_T)_2 \sqrt{v_2 - v_3} - \Lambda_1 \log_e \frac{\sqrt{v_D} + \sqrt{v_3}}{\sqrt{v_D} - \sqrt{v_3}}$$

$$- \int_0^{v_3} \left(\lambda_1 - \frac{\Lambda_1}{\sqrt{v_3 - v}} \right) \frac{dv}{\sqrt{v_2 - v}} + \int_{v_3}^{v_2} \frac{(\sigma_B - \sigma_T) - (\sigma_B - \sigma_T)_2}{2\sqrt{v_2 - v}} dv \quad (26)$$

$$\Lambda_1 = \frac{\sqrt{\frac{d}{du} v_3(u)}}{\pi} g_1(u_3, v) \quad (27)$$

and $(\sigma_B - \sigma_T)_2$ means the value of $(\sigma_B - \sigma_T)$ at curve 2. (See fig. 4.)

Calculation of surface velocities. - The velocity potential ϕ is given in u - v coordinates by the double integral

$$\varphi = -\frac{U}{\pi M} \int_0^{u_W} \int_0^{v_W} \frac{\sigma \, du \, dv}{\sqrt{(u_W-u)(v_W-v)}} \quad (28)$$

where u_W, v_W are the coordinates of a point on the wing and where σ is to be interpreted as the wing slope, or as the diaphragm slope, as required.

The velocities V_x and V_y are given by

$$\begin{aligned} V_x &= \frac{\partial \varphi}{\partial x} = \frac{M}{2\beta} \left(\frac{\partial \varphi}{\partial v_W} + \frac{\partial \varphi}{\partial u_W} \right) \\ V_y &= \frac{\partial \varphi}{\partial y} = \frac{M}{2} \left(\frac{\partial \varphi}{\partial v_W} - \frac{\partial \varphi}{\partial u_W} \right) \end{aligned} \quad (29)$$

The partial derivative $\frac{\partial \varphi}{\partial u_W}$ may be computed by use of the following equations, which are derived in appendix C as equations (C33) to (C36).

$$\frac{\partial \varphi}{\partial u_W} = \frac{U}{2M\pi} \left[\int_0^{u_W} \frac{R(u, v_W) - R(u_W, v_W)}{(u_W-u)^{3/2}} \, du - \frac{2R(u_W, v_W)}{\sqrt{u_W}} \right] \quad (30)$$

where, for points on the top surface of the wing,

$$\begin{aligned} R_T(u, v_W) &= 2\sigma_T(u, v_W) \sqrt{v_W - v_3} + \Lambda_1(u) \log_e \frac{\sqrt{v_W} + \sqrt{v_3}}{\sqrt{v_W} - \sqrt{v_3}} \\ &+ \int_0^{v_3} \left(\lambda_1 - \frac{\Lambda_1}{\sqrt{v_3-v}} \right) \frac{dv}{\sqrt{v_W-v}} \\ &+ \int_{v_3}^{v_W} \frac{\sigma_T(u, v) - \sigma_T(u, v_W)}{\sqrt{v_W-v}} \, dv \end{aligned} \quad (31)$$

for points on the bottom surface of the wing,

$$R_B(u, v_w) = 2\sigma_B(u, v_w) \sqrt{v_w - v_3} - \Lambda_1(u) \log_e \frac{\sqrt{v_w} + \sqrt{v_3}}{\sqrt{v_w} - \sqrt{v_3}} \\ - \int_0^{v_3} \left(\lambda_1 - \frac{\Lambda_1}{\sqrt{v_3 - v}} \right) \frac{dv}{\sqrt{v_w - v}} + \int_{v_3}^{v_w} \frac{\sigma_B(u, v) - \sigma_B(u, v_w)}{\sqrt{v_w - v}} dv \quad (32)$$

For points on the after diaphragm ($S_{D,4}$ in fig. 4),

$$R_T(u, v_D) = -R_B(u, v_D) = (\sigma_B - \sigma_T)_2 \left(\sqrt{v_D - v_3} - \sqrt{v_D - v_2} \right) \\ + \int_{v_3}^{v_2} \left[(\sigma_B - \sigma_T) - (\sigma_B - \sigma_T)_2 \right] \frac{dv}{\sqrt{v_D - v}} \quad (33)$$

and, for points on the forward diaphragm ($S_{D,1}$ in fig. 4), $R = 0$.

The function R is a first integral of the equation for the perturbation potential or for its derivative with respect to u_w . A similar function may be defined by interchanging the roles of u and v in equations (31) to (33) to obtain either the potential or its derivative with respect to v_w by a subsequent integration.

The paths of integration used in the calculation of $\frac{\partial \varphi}{\partial u_w}$ are shown in figure 7. Path 1 is used to compute the function R ; path 2, to compute φ or $\frac{\partial \varphi}{\partial u_w}$.

It is noted that R is singular at the point $v = v_3$ if $\Lambda_1 \neq 0$. This point does not appear in the integral for φ or $\frac{\partial \varphi}{\partial u_w}$ except when $v = v_3(u)$ and the singularity therefore introduces no difficulty because at $v = v_3(u)$ φ is finite and $\frac{\partial \varphi}{\partial u_w}$ is infinite if $\Lambda_1(u) \neq 0$.

The integrand of equation (30) is infinite at $u = u_w$. It is not convenient to remove this singularity because to do so would require the calculation of $\frac{\partial R(u, v_w)}{\partial u_w}$. Details of the results of integration by the use of a power-series expansion of $R(u, v_w) - R(u_w, v_w)$ in a small region of u near u_w are given in appendix C in a form suitable for numerical calculation.

As an illustration of the results of the numerical method, the diaphragm slopes and the pressure coefficients were computed for an unyawed flat-plate delta wing included within the Mach cone from the vertex. The locus of forward edges of the wings are specified in the oblique coordinates u and by the relations

$$v_3 = ku$$

$$v_2 = u/k \text{ or } u_2 = kv$$

In this case the diaphragm slopes are always subject to interaction effects. The calculations were carried out by a method of successive approximation. Eight equally spaced stations were selected from the forward Mach line to the edge of the wing along a line $u = \text{constant}$ on the diaphragm. Integrations were carried out by Simpson's rule.

The diaphragm slopes λ_1 for $k = 0.25, 0.50$, and 0.75 are shown in figure 8 together with the slopes neglecting all interaction as computed by means of equation (9). The dimensionless slopes λ/α have been multiplied by $\sqrt{1 - \frac{v}{ku}}$ in order to avoid representation of the pole at $\frac{v}{ku} = 1$. The intercept of the curve at $v/ku = 1$ is the value of Λ .

The results of this integration show that even for a sharply pointed wing ($k = 0.75$) the effect of the interaction is quite small. For many calculations it will therefore be permissible to neglect, at least for a first approximation, the effect of the interaction of external flow fields in the calculation of diaphragm slopes and pressure coefficients.

The pressure coefficients were computed by the use of values of R at eight equally spaced points along a line $v = \text{constant}$, with two additional points placed near the edge $v = u/k$. The exact pressure coefficients as computed by the methods of references 4 and 5 are compared with the numerically computed pressure

coefficients in figure 9. Agreement between the pressure coefficients as computed by the two methods is close, even though only a few points were used in the calculation.

Flight Propulsion Research Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, August 25, 1947.

APPENDIX A

SYMBOLS

The following symbols are used in this report:

A, B	coefficients
a, b	point values of function
C_p	pressure coefficient
$g(u, v)$	first integral for calculated diaphragm slopes
$g'(u, v) \equiv \frac{\partial g(u, v)}{\partial v}$	
$h'(u, v)$	regular terms of $g'(u, v)$
k	constant > 0
M	Mach number
$R(u, v)$	first integral for calculation of perturbation potential or its derivatives
S	plan-form area
U	free-stream velocity
u, v	oblique coordinates whose axes lie parallel to Mach lines
V	perturbation velocity
x, y, z	Cartesian coordinates (also used as subscripts to indicate components of velocity along coordinate axes)
α	angle of attack
$\beta \equiv \sqrt{M^2 - 1}$	cotangent of free-stream Mach angle
δ	interval of variable of integration
ξ, η	Cartesian coordinates

λ	slope of stream sheet near plane of wing measured in $y = \text{constant}$ planes
Λ	strength of singularity in diaphragm slope at wing edge
σ	slope of wing surface with respect to x, y plane as measured in $y = \text{constant}$ planes
ϕ	perturbation velocity potential

Subscripts:

B	bottom wing surface
D	diaphragm
T	top wing surface
w	wing

1, 2, 3, etc. refers to numbered curves or surface areas

Examples:

σ_T	slope of top surface of wing
$\phi_{T,D}$	potential on top surface of wing due to diaphragm
$S_{w,3}$	wing area 3
v_1	curve $v = v_1(u)$
u_1	curve $u = u_1(v)$
λ_1	slope of diaphragm in plan area 1
$(\sigma_B - \sigma_T)_2$	difference between slopes of bottom and top wing surface at curve 2
$g_{2,D}(v_D)$	portion of function $g_2(v_D)$ due to diaphragm elements lying ahead of the wing on a line $u = \text{constant}$
$R_T(u, v_w)$	first integral of perturbation potential for points on wing having the coordinate v_w

APPENDIX B

EVALUATION OF EQUATION (6) FOR WING WHOSE PROFILE

IS SYMMETRICAL ABOUT PLANE OF WING

In this case, $\frac{\sigma_B - \sigma_T}{2} = \alpha$, the angle of attack; thus equation (6) becomes

$$\lambda(u, z) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \int_{v_1(u)}^{v_2(u)} \frac{\alpha dv}{\sqrt{v_D-v}} \quad (B1)$$

Evaluation of the inside integral yields

$$\lambda(u, z) = -\frac{2\alpha}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{(\sqrt{v_D-v_2} - \sqrt{v_D-v_1})}{\sqrt{z-v_D}} dv_D \quad (B2)$$

The two integrations indicated in equation (B2) may be obtained from formulas 111 and 113 of reference 6 to give

$$-\int_{v_2(u)}^z \frac{\sqrt{v_D-v_2}}{\sqrt{z-v_D}} dv_D = \frac{\pi}{2} (z-v_2) \quad (B3)$$

$$\int_{v_2(u)}^z \frac{\sqrt{v_D-v_1}}{\sqrt{z-v_D}} dv_D = \sqrt{(z-v_1)(v_2-v_1)} + (z-v_1) \tan^{-1} \sqrt{\frac{z-v_2}{v_2-v_1}} \quad (B4)$$

Substitution of equations (B3) and (B4) into equation (B2) yields

$$\lambda(u, z) = -\frac{2\alpha}{\pi} \frac{\partial}{\partial z} \left[\frac{\pi}{2} (z-v_2) - \sqrt{(z-v_2)(v_2-v_1)} - (z-v_1) \tan^{-1} \sqrt{\frac{z-v_2}{v_2-v_1}} \right]$$

from which

$$\lambda(u, v) = \frac{2\alpha}{\pi} \left(\tan^{-1} \sqrt{\frac{v-v_2}{v_2-v_1}} + \sqrt{\frac{v_2-v_1}{v-v_2}} - \frac{\pi}{2} \right) = \frac{2\alpha}{\pi} \left(\sqrt{\frac{v_2-v_1}{v-v_2}} - \tan^{-1} \sqrt{\frac{v_2-v_1}{v-v_2}} \right) \quad (B5)$$

That equation (B5) is a solution of equation (3) may be verified by substitution:

$$\int_{v_2(u)}^{v_D} \frac{\lambda(u, v) dv}{\sqrt{v_D-v}} = \frac{2\alpha}{\pi} \int_{v_2(u)}^{v_D} \frac{1}{\sqrt{v_D-v}} \tan^{-1} \sqrt{\frac{v-v_2}{v_2-v_1}} dv + \frac{2\alpha}{\pi} \int_{v_2(u)}^{v_D} \sqrt{\frac{(v_2-v_1)}{(v-v_2)(v_D-v)}} dv - \alpha \int_{v_2(u)}^{v_D} \frac{dv}{\sqrt{v_D-v}} \quad (B6)$$

The first integral of the second member may be integrated by parts to give

$$\begin{aligned} \int_{v_2}^{v_D} \frac{1}{\sqrt{v_D-v}} \tan^{-1} \sqrt{\frac{v-v_2}{v_2-v_1}} dv &= -2 \sqrt{v_D-v} \tan^{-1} \sqrt{\frac{v-v_2}{v_2-v_1}} \Big|_{v_2}^{v_D} \\ &+ \int_{v_2}^{v_D} \frac{\sqrt{(v_D-v)(v_2-v_1)}}{(v-v_1) \sqrt{v-v_2}} dv = \int_{v_2}^{v_D} \frac{\sqrt{(v_D-v)(v_2-v_1)}}{(v-v_1) \sqrt{v-v_2}} dv \end{aligned} \quad (B7)$$

Combination of the second integral of equations (B6) and (B7) yields

$$\begin{aligned} & \frac{2\alpha}{\pi} \int_{v_2(u)}^{v_D} \left[\frac{1}{\sqrt{v_D-v}} \tan^{-1} \sqrt{\frac{v-v_2}{v_2-v_1}} + \sqrt{\frac{(v_2-v_1)}{(v-v_2)(v_D-v)}} \right] dv \\ &= \frac{2\alpha}{\pi} \sqrt{(v_2-v_1)(v_D-v_1)} \int_{v_2}^{v_D} \frac{dv}{(v-v_1)\sqrt{-v^2+(v_D+v_2)v-v_2v_D}} \quad (B8) \end{aligned}$$

Equation (B8) was integrated by formula 195 of reference 6 to give

$$\frac{2\alpha}{\pi} \int_{v_2(u)}^{v_D} \left[\frac{1}{\sqrt{v_D-v}} \tan^{-1} \sqrt{\frac{v-v_2}{v_2-v_1}} + \sqrt{\frac{v_2-v_1}{(v-v_2)(v_D-v)}} \right] dv = 2\alpha \sqrt{v_D-v_1} \quad (B9)$$

Evaluation of the third integral of equation (B6) yields

$$-\alpha \int_{v_2(u)}^{v_D} \frac{dv}{\sqrt{v_D-v}} = -2\alpha \sqrt{v_D-v_2} \quad (B10)$$

Substitution of equations (B9) and (B10) into equation (B6) yields

$$\int_{v_2(u)}^{v_D} \frac{\lambda(u,v) dv}{\sqrt{(v_D-v)}} = 2\alpha \left(\sqrt{v_D-v_1} - \sqrt{v_D-v_2} \right) \quad (B11)$$

But

$$\int_{v_1(u)}^{v_2(u)} \frac{\alpha dv}{\sqrt{v_D-v}} = 2\alpha \left(\sqrt{v_D-v_1} - \sqrt{v_D-v_2} \right) \quad (B12)$$

A comparison of equations (B11) and (B12) shows that λ as given by equation (B5), is a solution of the equation

$$\int_{v_2(u)}^{v_D} \frac{\lambda(u, v) dv}{\sqrt{(v_D - v)}} = \int_{v_1(u)}^{v_2(u)} \frac{\alpha dv}{\sqrt{v_D - v}} \quad (B13)$$

APPENDIX C

NUMERICAL CALCULATION OF DIAPHRAGM SLOPES AND VELOCITIES

In order to compute diaphragm slopes and velocities or pressure coefficients by numerical methods, all the infinite singularities with the integrals must be removed and evaluated separately. Whenever possible, numerical differentiation should be eliminated. It is also possible to evaluate integrals having infinite singularities by expanding the integrand in series near the pole and then evaluating the integral by term-wise integration of the function.

The first method has been chosen because in many cases, for example, for wings with flat surfaces, the integrand left after removal of the pole vanishes for all or part of the region of integration.

The first part of this discussion in appendix C is devoted to the isolation of singularities in the integrals defining λ . The second part considers methods of computing velocities on the wing surface.

Calculation of λ . - The diaphragm slope λ is in general given by an equation of the form of equation (20) or equation (22a), which for the region aft a wing surface in the direction of increasing v is

$$\lambda_2(u, z) = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{v_2(u)}^z \frac{dv_D}{\sqrt{z-v_D}} \left[\int_{v_3(u)}^{v_2(u)} \frac{\sigma_B - \sigma_T}{\sqrt{v_D-v}} dv - \int_0^{v_3(u)} \frac{\lambda_1(u, v) dv}{\sqrt{v_D-v}} \right] \quad (C1)$$

evaluated at $z = v$, where λ_1 is the diaphragm slope ahead of the wing on a line $u = \text{constant}$.

Equation (C1) may be formally simplified by defining the function $g_2(v_D)$

$$g_2(v_D) = \int_{v_3(u)}^{v_2(u)} \frac{\sigma_B - \sigma_T}{2\sqrt{v_D-v}} dv - \int_0^{v_3(u)} \frac{\lambda_1 dv}{\sqrt{v_D-v}} \quad (C2)$$

The calculation of λ_2 may be divided into two parts: the calculation of $\lambda_{2,w}$ due to the function

$$g_{2,w}(v_D) = \int_{v_3(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T)_2 dv}{2\sqrt{v_D - v}} \quad (C3)$$

and the calculation of $\lambda_{2,D}$ due to the function

$$g_{2,D}(v_D) = \int_0^{v_3(u)} \frac{\lambda_1 dv}{\sqrt{v_D - v}} \quad (C4)$$

Each of the integrands of equations (C3) and (C4) may contain poles of order 1/2 for some value of v . The integrand defining $g_{2,w}$ is singular for $v_D = v_2(u)$. The effect of the singularity may be isolated and evaluated by

$$\begin{aligned} g_{2,w}(v_D) &= \int_{v_3(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T)_2 (\sigma_B - \sigma_T)_2 dv}{2\sqrt{v_D - v}} + \int_{v_3(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T)_2 dv}{2\sqrt{v_D - v}} \\ &= (\sigma_B - \sigma_T)_2 \left(\sqrt{v_D - v_3(u)} - \sqrt{v_D - v_2(u)} \right) \\ &\quad + \int_{v_3(u)}^{v_2(u)} \frac{[(\sigma_B - \sigma_T)_2 - (\sigma_B - \sigma_T)_2] dv}{2\sqrt{v_D - v}} \end{aligned} \quad (C5)$$

where the term $(\sigma_B - \sigma_T)_2$ is used to mean the value $\Lambda(\sigma_B - \sigma_T)$ at curve 2 and in which the second term has a finite integrand even at $v_D = v_2$.

The integrand defining $\varepsilon_{2,D}(v_D)$ has a pole due to λ_1 of order $1/2$ at $v_D = v_3(u)$. The effect of this singularity may be isolated by

$$\lambda_1(v) = \frac{\Lambda_1}{\sqrt{v_3(u)-v}} + \mu(v) \quad (C6)$$

and by rewriting equation (C4) in the form

$$\varepsilon_{2,D}(v_D) = \int_0^{v_3(u)} \frac{-\Lambda_1 dv}{\sqrt{[v_3(u)-v](v_D-v)}} + \int_0^{v_3(u)} \frac{-\mu(v)dv}{\sqrt{v_D-v}} \quad (C7)$$

If the value of Λ_1 is computed by the use of the equation

$$\Lambda_1 = \lim_{v \rightarrow v_3(u)} \sqrt{v_3(u)-v} \lambda_1 \quad v \leq v_1(u) \quad (C8)$$

the function $\mu(v)$ will be everywhere finite, as may be seen for a special case from equation (9). The function λ_1 is defined by the equation

$$\lambda_1 = \frac{1}{\pi} \frac{\partial}{\partial z} \int_{u_3(v)}^z \frac{du_D}{\sqrt{z-u_D}} \int_0^{u_3(v)} \frac{(\sigma_B - \sigma_T) du}{2\sqrt{u_D-u}} \quad (C9)$$

evaluated at $z = u$.

The term $(\sigma_B - \sigma_T)/2$ in equations (C9) and (C10) is to be interpreted as including $-\lambda_2$ if the integration from 0 to $u_3(v)$ includes a portion of the area $S_{D,2}$.

In the immediate vicinity of the edge $u_3(v)$, the function λ_1 has a value due mainly to the strength of the pole at that edge. As measured along a line $v = \text{constant}$, in the vicinity of the edge $\sqrt{u-u_D}$ in equation (C9) can be replaced by its mean value, in which case equation (C9) reduces to the form

$$\lambda_1 \approx \frac{1}{\pi \sqrt{u - u_3(v)}} \int_0^{u_3(v)} \frac{(\sigma_B - \sigma_T) du}{2 \sqrt{u_3(v) - u}} \quad (C10)$$

In all cases, except for an airfoil with a pointed tip having both edges swept behind the Mach angle, the value of the integral

$$\int_0^{u_3(v)} \frac{(\sigma_B - \sigma_T) du}{2 \sqrt{u_D - u}} = g_1(u_D) \quad (C11)$$

(for example, see equation (9)), or the value of the equivalent expression in v will be known from a previous calculation of $g_1(u_D)$ or $g_2(v)$. In the vicinity of the curve $u = u_3(v)$, equation (C10) may therefore be written as

$$\lambda_1 \approx \frac{g_1(u_3)}{\pi \sqrt{u - u_3}} \quad (C12)$$

From equations (C8) and (C12)

$$\Lambda_1 = \lim_{v \rightarrow v_3(u)} \frac{g_1(u_3)}{\pi} \sqrt{\frac{v_3 - v}{u - u_3}} = \frac{g_1(u_3)}{\pi} \sqrt{\frac{\partial}{\partial u} v_3} \quad (C13)$$

The contribution to $g_{2,D}(v_D)$ from Λ_1 is

$$- \int_0^{v_3(u)} \frac{\Lambda_1 dv}{\sqrt{(v_3 - v)(v_D - v)}} = - \Lambda_1 \log_e \frac{\sqrt{v_D} + \sqrt{v_3}}{\sqrt{v_D} - \sqrt{v_3}} \quad (C14)$$

hence, $g_{2,D}(v_D)$ may be written as

$$g_{2,D}(v_D) = - \int_0^{v_3} \left(\lambda_1 - \frac{\Lambda_1}{\sqrt{v_3 - v}} \right) \frac{dv}{\sqrt{v_D - v}} - \Lambda_1 \log_e \frac{\sqrt{v_D} + \sqrt{v_3}}{\sqrt{v_D} - \sqrt{v_3}} \quad (C15)$$

Finally, $g_2(v_D)$ is given by the relations

$$\begin{aligned}
 g_2(v_D) = & (\sigma_B - \sigma_T)_2 \left(\sqrt{v_D - v_3} \cdot \sqrt{v_D - v_2} \right) \\
 & - \Lambda_1 \log_e \frac{\sqrt{v_D} + \sqrt{v_3}}{\sqrt{v_D} - \sqrt{v_3}} + \int_{v_3}^{v_2(u)} \frac{[(\sigma_B - \sigma_T) - (\sigma_B - \sigma_T)_2] dv}{2 \sqrt{v_D - v}} \\
 & - \int_0^{v_3} \left(\lambda_1 - \frac{\Lambda_1}{\sqrt{v_3 - v}} \right) \frac{dv}{\sqrt{v_D - v}}
 \end{aligned} \tag{C16}$$

where

$$\Lambda_1 = \frac{\sqrt{\frac{\partial}{\partial u} v_3}}{\pi} g_1(u_3) \tag{C13}$$

is evaluated at curve 3,

For a purely numerical integration of the equations for λ , it is desirable to eliminate the differentiation outside the integral sign in equation (C1). The diaphragm slope λ_2 can be computed by the use of the equation

$$\begin{aligned}
 \lambda_2(u, v) = & \frac{1}{\pi} \frac{\partial}{\partial v} \int_{v_2}^v \frac{dv_D}{\sqrt{v - v_D}} g_2(v_D) \\
 = & \frac{1}{\pi} \frac{g_2(v_2)}{\sqrt{v - v_2}} + \frac{1}{\pi} \int_{v_2}^v \frac{g_2'(v_D) dv_D}{\sqrt{v - v_D}}
 \end{aligned} \tag{C17}$$

which may be derived by integration by parts prior to the differentiation. The quantity $g_2'(v_D)$ is defined as

$$g_2'(v_D) = \frac{\partial}{\partial v_D} g_2(v_D)$$

Equation (C17) is valid if $g'(v_D)$ is finite within the interval $v_2(u)$ to v and at its limits and has at most a finite number of discontinuities. In order to use equation (C17), the portions of $g'(v)$ that do not satisfy these conditions must be isolated and evaluated separately.

The integrand of equation (C17) is always singular at the point $v = v_D$.

In general $g_2'(v_D)$ also has a pole of order $1/2$ at $v = v_2(u)$ contributed by the term $(\sigma_B - \sigma_T)_2 \sqrt{v_D - v_2}$ in equation (C16). Its contribution to λ_2 is

$$- \frac{1}{2} (\sigma_B - \sigma_T)_2 \quad (C18)$$

The integral in the general expression for $g_2'(v_D)$ due to the integral

$$\int_{v_3}^{v_2} \frac{(\sigma_B - \sigma_T) - (\sigma_B - \sigma_T)_2}{2\sqrt{v_D - v}} dv \quad (C19)$$

in equation (C16) will have an infinite integrand at $v_D = v_2(u)$, unless the term

$$\left[\frac{\partial(\sigma_B - \sigma_T)}{\partial v} \right]_2 = 0 \quad (C20)$$

The expression (C19) may be written

$$\begin{aligned}
& \int_{v_3}^{v_2} \frac{(\sigma_B - \sigma_T) - (\sigma_B - \sigma_T)_2 - (v - v_2) \left[\frac{\partial(\sigma_B - \sigma_T)}{\partial v} \right]_2}{2\sqrt{v_D - v}} dv \\
& + \int_{v_3}^{v_2} \frac{(v - v_2)}{2\sqrt{v_D - v}} \left(\frac{\partial(\sigma_B - \sigma_T)}{\partial v} \right)_2 dv \quad (C21)
\end{aligned}$$

The first term of expression (C21) has an everywhere finite derivative with a finite integrand given by

$$\int_{v_3}^{v_2} \frac{(\sigma_B - \sigma_T) - (\sigma_B - \sigma_T)_2 - (v - v_2) \left[\frac{\partial(\sigma_B - \sigma_T)}{\partial v} \right]_2}{4(v_D - v)^{3/2}} dv \quad (C22)$$

The contribution of the second term of expression (C21) to $g(v_D)$ is

$$\begin{aligned}
& \frac{1}{3} \left[\frac{\partial(\sigma_B - \sigma_T)}{\partial v} \right]_2 \left[2(v_D - v_2) (\sqrt{v_D - v_3} + \sqrt{v_D - v_2}) \right. \\
& \left. - (v_2 - v_3) \sqrt{v_D - v_3} \right] \quad (C23)
\end{aligned}$$

for which the corresponding contribution to $g_2'(v_D)$ is

$$\left[\frac{\partial(\sigma_B - \sigma_T)}{\partial v} \right]_2 \left[\frac{(v_D - v_3) + (v_D - v_2)}{2\sqrt{v_D - v_3}} - \sqrt{v_D - v_2} \right]$$

Finally, λ_2 may be computed by adding the contributions of the regular and singular terms of $g_2(v_D)$ and $g_2'(v_D)$:

$$\lambda_2(u, v) = \frac{1}{\pi} \frac{g_2(u, v_2)}{\sqrt{v-v_2}} - \frac{1}{2} (\sigma_B - \sigma_T)_2 + \frac{1}{\pi} \int_{v_2(u)}^v \frac{h'(v_D) - h'(v)}{\sqrt{v-v_D}} dv_D + \frac{2}{\pi} h'(v) \sqrt{v-v_2} \quad (C24)$$

where

$$h'(v_D) = \frac{(\sigma_B - \sigma_T)_2}{2\sqrt{v_D - v_3}} + \Lambda_1 \frac{\sqrt{\frac{v_3}{v_D}}}{v_D - v_3} + \left[\frac{(v_D - v_3) + (v_D - v_2)}{2\sqrt{v_D - v_3}} - \sqrt{v_D - v_2} \right] \left[\frac{\partial(\sigma_B - \sigma_T)}{\partial v} \right]_2 + \frac{1}{2} \int_0^{v_3} \left(\lambda_1 - \frac{\Lambda_1}{\sqrt{v_3 - v}} \right) \frac{dv}{(v_D - v)^{3/2}} - \int_{v_3(u)}^{v_2(u)} \frac{(\sigma_B - \sigma_T) - (\sigma_B - \sigma_T)_2 - (v - v_2) \left[\frac{\partial(\sigma_B - \sigma_T)}{\partial v} \right]_2}{4(v_D - v)^{3/2}} dv \quad (C25)$$

The function $h'(v_D)$ contains all the regular terms of $g_2'(v_D)$.

Calculation of surface velocities. - The velocity potential φ is given by the double integral

$$\varphi = -\frac{U}{\pi} \iint \frac{\sigma d\xi d\eta}{\sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2}} \quad (C26)$$

in which the integration is carried out over the forward Mach cone or over limited regions of this cone as described in reference (1) and where σ represents the slope of the wing surface or the slope of the diaphragm, as required.

In most cases interest centers in the velocities or in the pressure coefficients, which are proportional to the x component of the velocity. The components of velocity are given by the equations

$$v_x = \frac{\partial \phi}{\partial x} = -\frac{U}{\pi} \frac{\partial}{\partial x} \iint \frac{\sigma d\xi d\eta}{\sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2}}$$

$$v_y = \frac{\partial \phi}{\partial y} = -\frac{U}{\pi} \frac{\partial}{\partial y} \iint \frac{\sigma d\xi d\eta}{\sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2}} \quad (C27)$$

The manipulation of these derivatives is facilitated by transforming the integrals to the u, v coordinates.

$$\phi(u_w, v_w) = -\frac{U}{M\pi} \int_0^{u_w} \int_0^{v_w} \frac{\sigma du dv}{\sqrt{(u_w-u)(v_w-v)}} \quad (C28)$$

where u_w and v_w are the coordinates of a point on the wing. In terms of equations (C27) and (C28)

$$v_x = \frac{\partial \phi}{\partial x} = \frac{M}{2\beta} \left(\frac{\partial \phi}{\partial u_w} + \frac{\partial \phi}{\partial v_w} \right)$$

$$v_y = \frac{\partial \phi}{\partial y} = \frac{M}{2} \left(\frac{\partial \phi}{\partial v_w} - \frac{\partial \phi}{\partial u_w} \right) \quad (C29)$$

The derivatives of the potential with respect to u_w and v_w are given by the relations

$$\frac{\partial \phi}{\partial u_W} = \frac{-U}{M\pi} \frac{\partial}{\partial u_W} \int_0^{u_W} \int_0^{v_W} \frac{\sigma du dv}{\sqrt{(u_W-u)} \sqrt{v_W-v}}$$

$$\frac{\partial \phi}{\partial v_W} = \frac{-U}{M\pi} \frac{\partial}{\partial v_W} \int_0^{u_W} \int_0^{v_W} \frac{\sigma du dv}{\sqrt{u_W-u} \sqrt{(v_W-v)}} \quad (C30)$$

The calculation of the derivatives may be simplified in the manner subsequently described for the calculation of $\partial \phi / \partial u_W$. The equation for the potential may be written as

$$\phi = -\frac{U}{M\pi} \int_0^{u_W} \frac{du}{\sqrt{u_W-u}} \int_0^v \frac{\sigma(u,v)}{\sqrt{v_W-v}} dv \quad (C31)$$

where the symbol v_W is understood to include v_D when the limit of integration lies on one of the diaphragms, in which case the contribution of σ from both top and bottom wing surfaces forward of v_D must be considered.

It is convenient to define a function $R(u, v_W)$ by the relation

$$R(u, v_W) = \int_0^{v_W} \frac{\sigma(u, v) dv}{\sqrt{v_W-v}} \quad (C32)$$

then

$$\frac{\partial \phi}{\partial u_W} = \frac{U}{2M\pi} \left[\int_0^{u_W} \frac{R(u, v_W) - R(u_W, v_W)}{(u_W-u)^{3/2}} du - 2 \frac{R(u_W, v_W)}{\sqrt{u_W}} \right] \quad (C33)$$

For points on the top surface of the wing $R_T(u, v_W)$ is given by the relation

$$\begin{aligned}
R_T(u, v_W) = & 2\sigma_T(u, v_W) \sqrt{v_W - v_3} + \Lambda_1(u) \log_e \frac{\sqrt{v_W} + \sqrt{v_3}}{\sqrt{v_W} - \sqrt{v_3}} \\
& + \int_0^{v_3} \left[\lambda_1 - \frac{\Lambda_1(u)}{\sqrt{v_3 - v}} \right] \frac{dv}{\sqrt{v_W - v}} \\
& + \int_{v_3}^{v_W} \frac{\sigma_T(u, v) - \sigma_T(u, v_W)}{\sqrt{v_W - v}} dv \quad (C34)
\end{aligned}$$

and the value of $R_B(u, v_W)$ is given by the relation

$$\begin{aligned}
R_B(u, v_W) = & 2\sigma_B(u, v_W) \sqrt{v_W - v_3} - \Lambda_1(u) \log_e \frac{\sqrt{v_W} + \sqrt{v_3}}{\sqrt{v_W} - \sqrt{v_3}} \\
& - \int_0^{v_3} \left[\lambda_1 - \frac{\Lambda_1(u)}{\sqrt{v_3 - v}} \right] \frac{dv}{\sqrt{v_W - v}} \\
& + \int_{v_3}^{v_W} \frac{\sigma_B(u, v) - \sigma_B(u, v_W)}{\sqrt{v_W - v}} dv \quad (C35)
\end{aligned}$$

For a point on the after diaphragm (λ_2), the value of $R_T(u, v_D)$ and $-R_B(u, v_D)$ is

$$\begin{aligned}
R_T(u, v_D) = -R_B(u, v_D) = & 2(\sigma_B + \sigma_T)_2 \left(\sqrt{v_D - v_3} - \sqrt{v_D - v_2} \right) \\
& + \int_{v_3(u)}^{v_2(u)} \left[(\sigma_B + \sigma_T) - (\sigma_B + \sigma_T)_2 \right] \frac{dv}{\sqrt{v_D - v}} \quad (C36)
\end{aligned}$$

For the calculation of lift distributions, the contribution of $R_T(u, v_D)$ and $R_B(u, v_D)$ cancel and may therefore be neglected.

For flat-plate wings $\sigma_B + \sigma_T = 0$.

For points on the Forward diaphragm (λ_1), the defining integrals for $R(u, v_D)$ includes only the effect of λ_1 . The value of R is therefore zero.

The integrand of the integral in equation (C33) generally becomes infinite at $u = u_w$. Although this integral could be integrated by parts and the integral obtained in terms of the derivative of $R(u_w, v_w)$ with respect to u_w , it seems preferable in this case to expand $R(u, v_w)$ in a power series about the point u_w . When the function $R(u, v_w)$ can be expanded with sufficient accuracy in a segment of width 2δ

$$u_w \geq u \geq u_w - 2\delta$$

as a parabola of the form

$$R(u, v_w) = R(u_w, v_w) + A(u_w - u) + B(u_w - u)^2$$

and if

$$R(u_w - 2\delta, v_w) - R(u_w, v_w) = a$$

$$R(u_w - \delta, v_w) - R(u_w, v_w) = b$$

it may be easily shown that

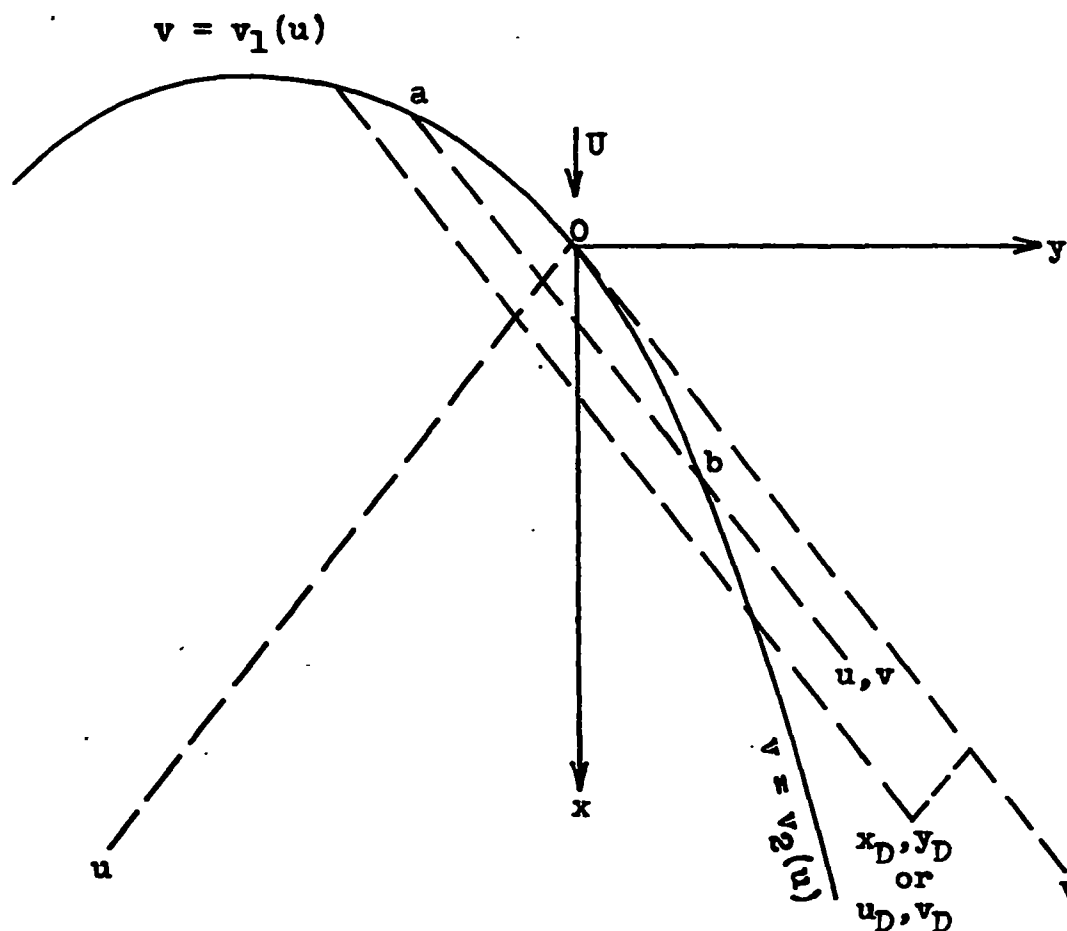
$$\int_{u_w - 2\delta}^{u_w} \frac{R(u, v_w) - R(u_w, v_w)}{(u_w - u)^{3/2}} du = \sqrt{\frac{2}{\delta}} \frac{8b - a}{3}$$

It is of interest to note that the function R is finite for all values of u except $u = u_3(v)$. $R(u_3, v_w)$ and $\frac{\partial \varphi}{\partial u_w}$ are finite at the point (u_3, v_w) if $\Lambda_1(u_3, v_3)$ is zero.

The calculation of $\frac{\partial \varphi}{\partial v_w}$ is carried out similarly to that of $\frac{\partial \varphi}{\partial u_w}$, except that the roles of u and v are interchanged and a suitable first integral is defined to take the place of $R(u, v_w)$.

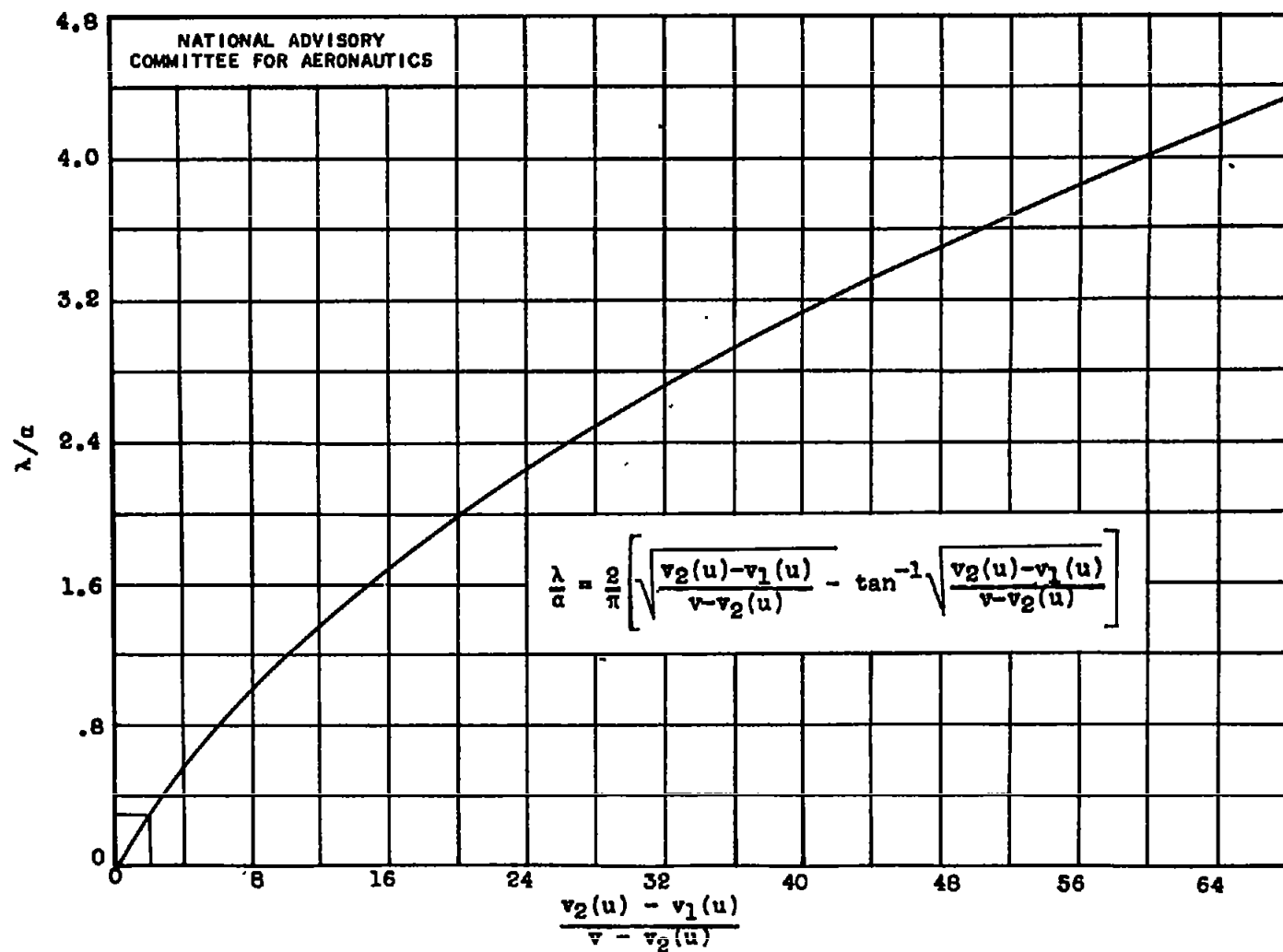
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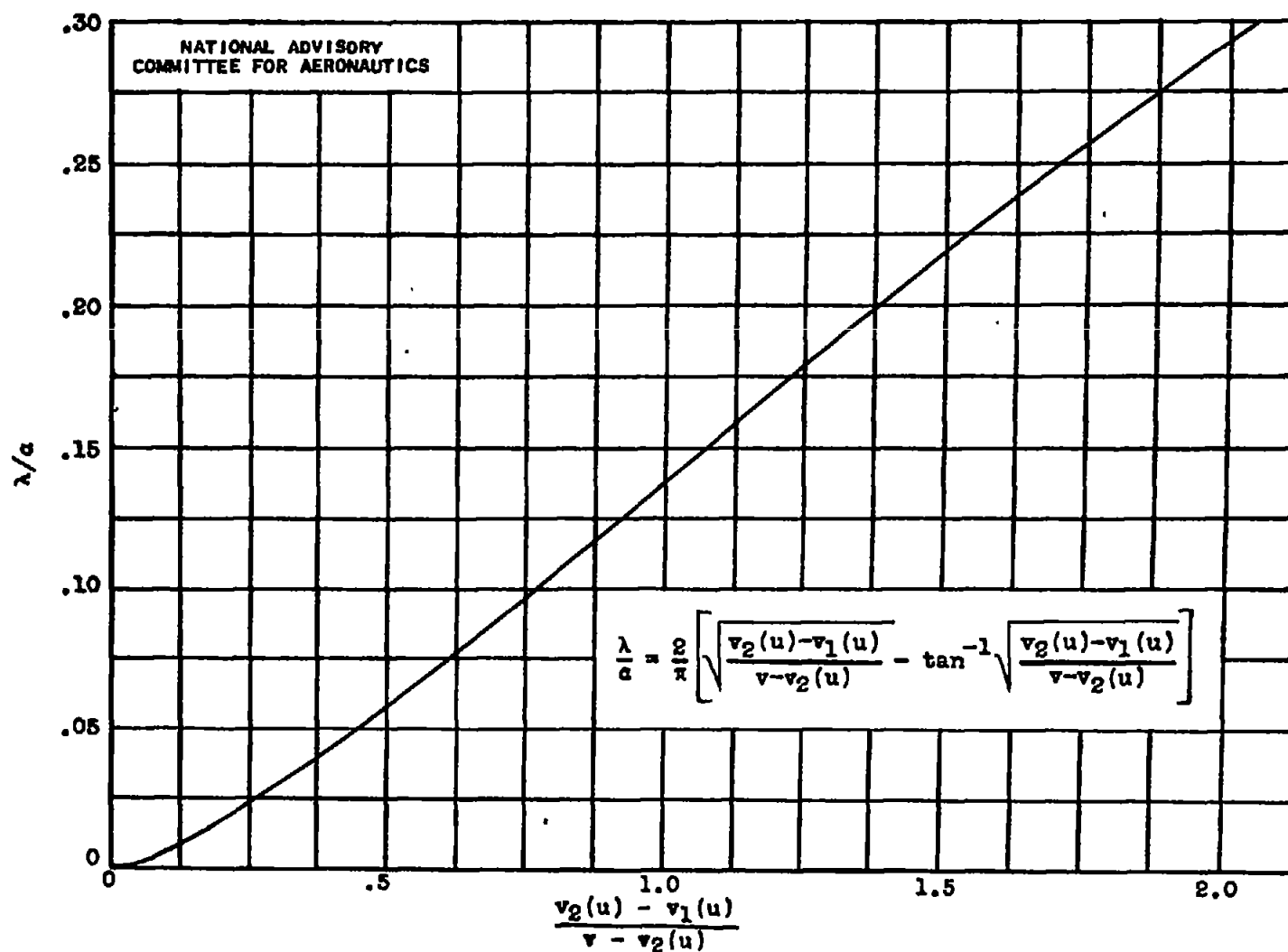
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Figure 1. - Wing plan form and diaphragm for equation (3).



(a) Over-all plot.

Figure 2. - Variation of diaphragm slope to wing angle-of-attack ratio for wedge-profile wing.



(b) Enlargement of area shown in figure 2(a).

Figure 2. - Concluded. Variation of diaphragm slope to wing angle-of-attack ratio for wedge-profile wing.

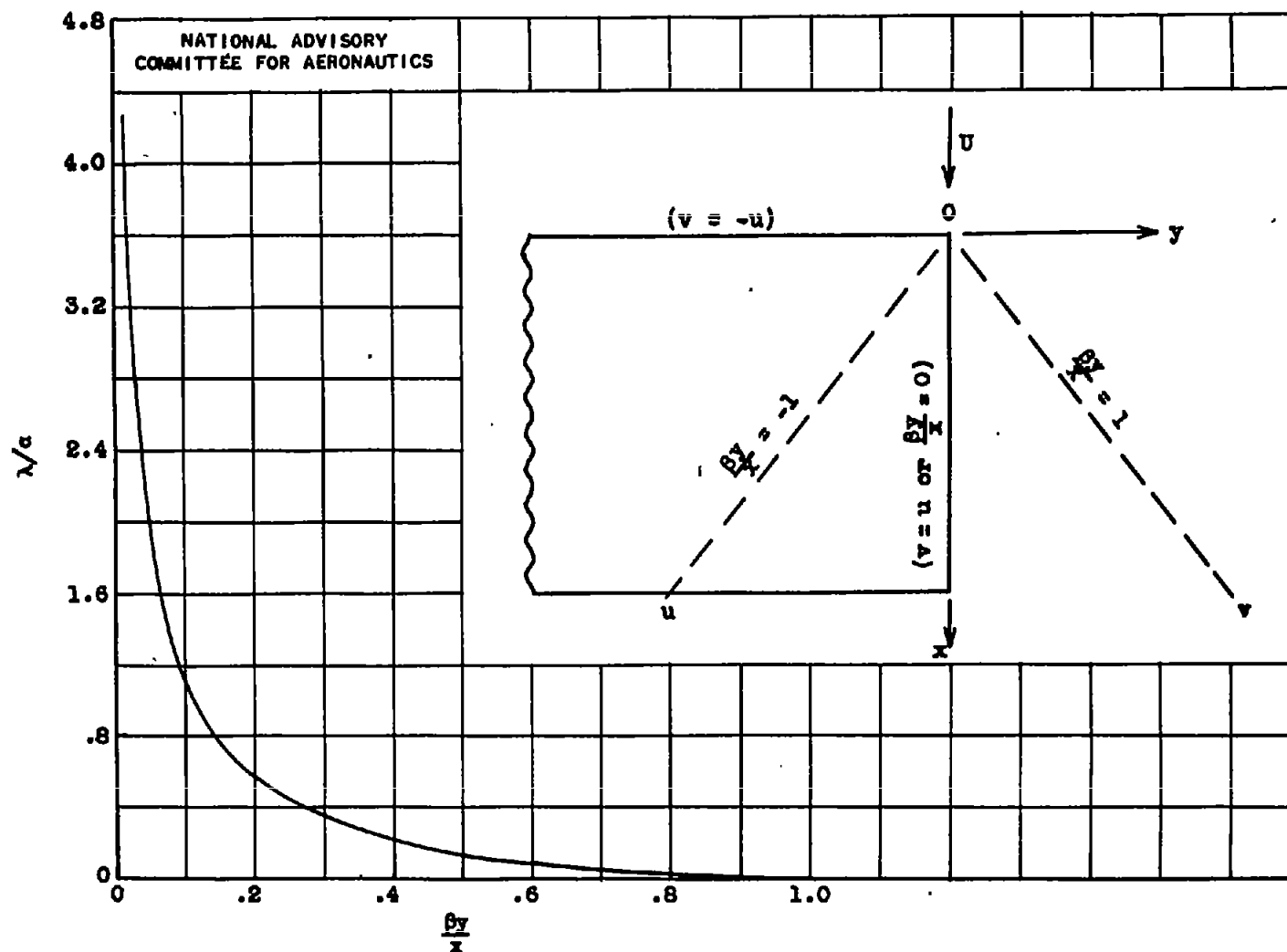


Figure 3. - Variation of diaphragm slope to wing angle-of-attack ratio for rectangular plan-form thin flat-plate wing.

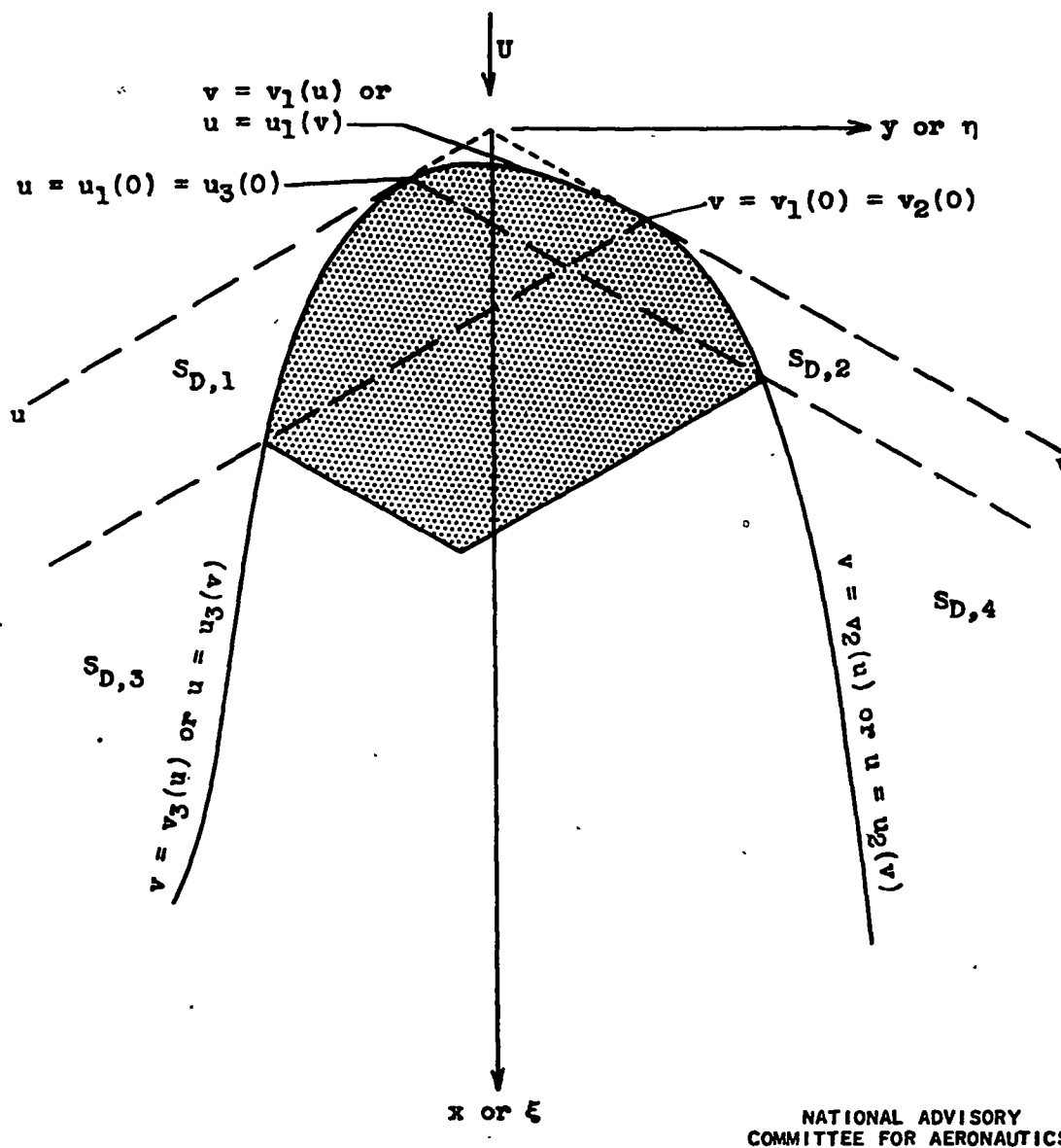
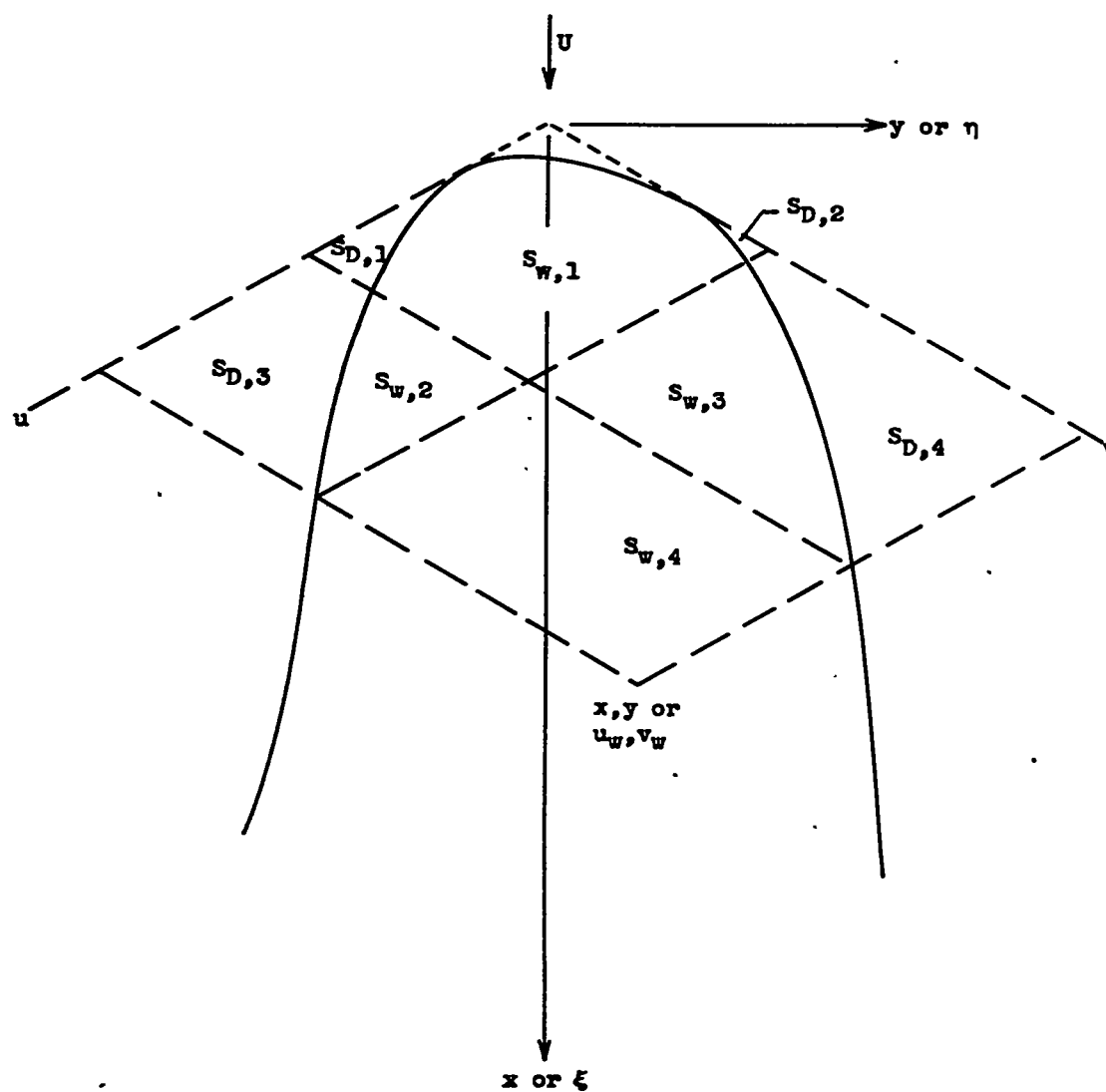
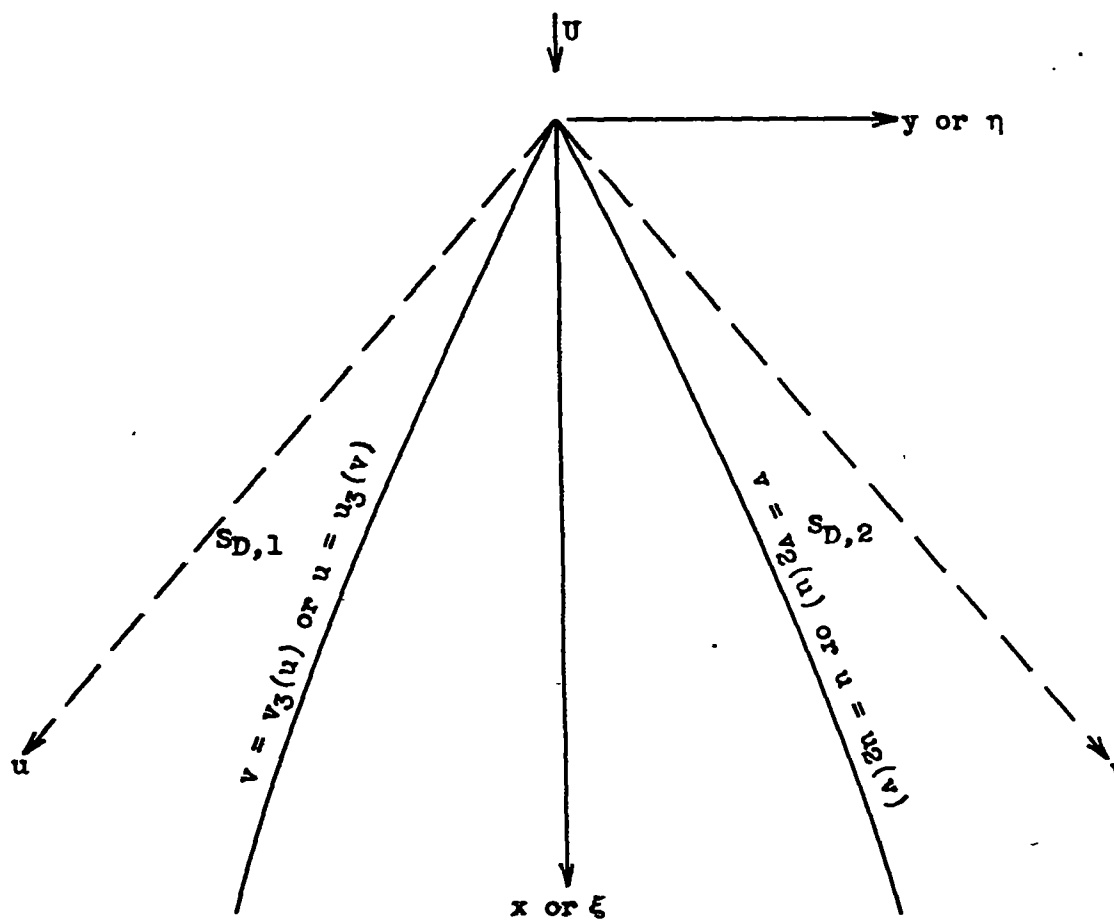


Figure 4. - Wing plan form and diaphragm areas for equations (11) to (14). (Velocity potential of shaded region may be calculated by methods of reference 1.)



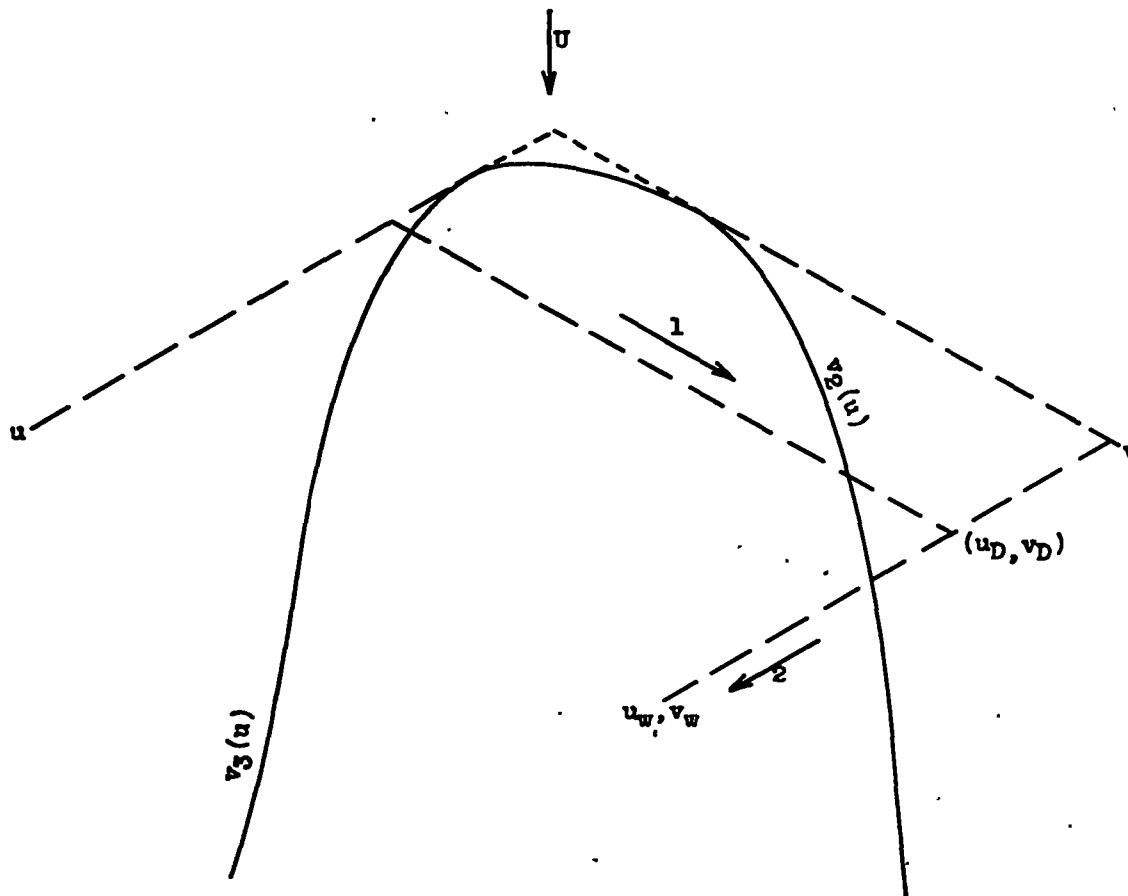
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Figure 5. - Regions of integration for equations (15) to (19).



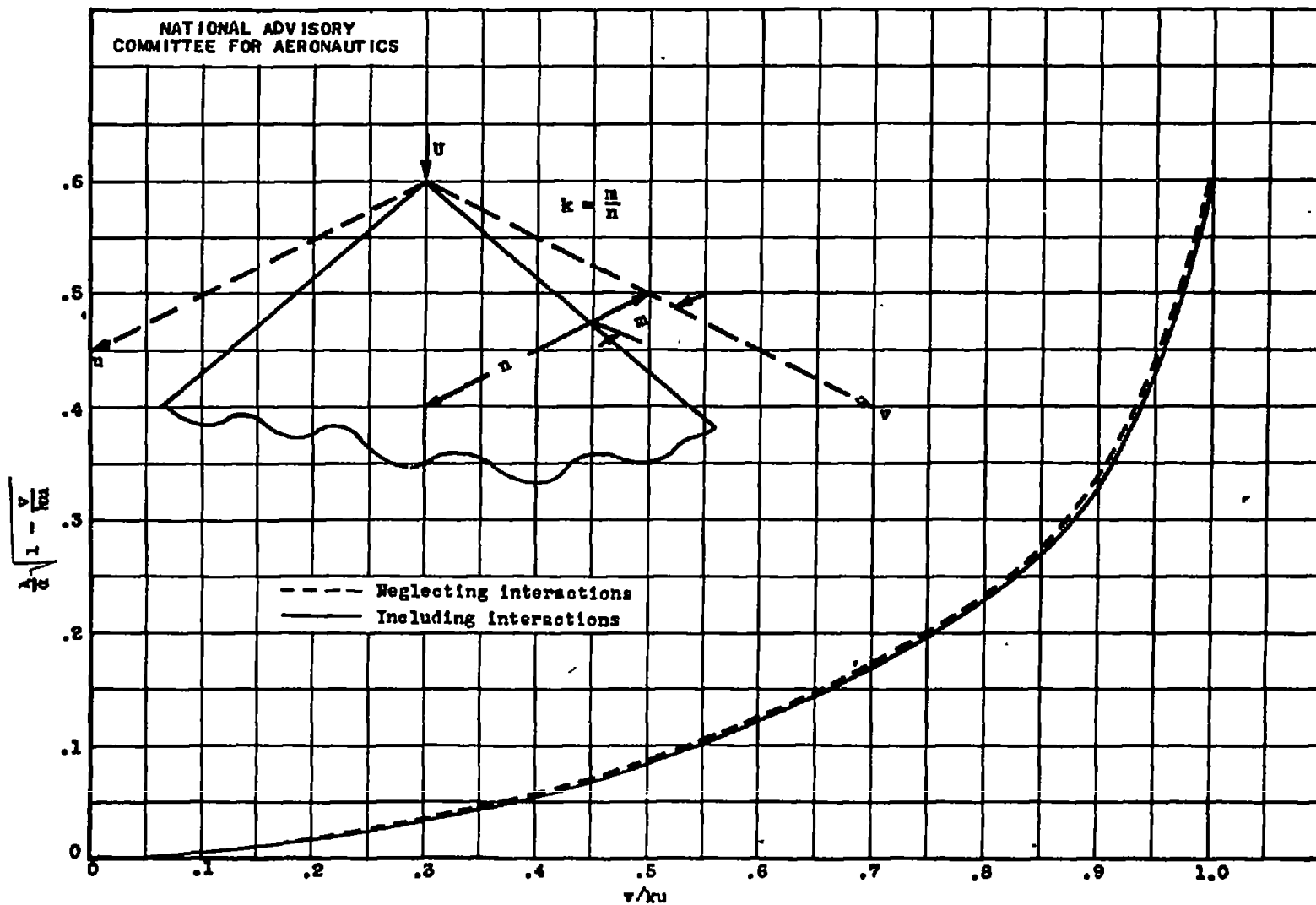
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Figure 6. - Wing plan form and diaphragm areas for equations (17) and (18).



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Figure 7. - Paths of integration used in computing perturbation velocities on wing.



(a) $k = 0.25$.

Figure 8. - Diaphragm slopes λ/α for symmetrical flat-plate delta wing.

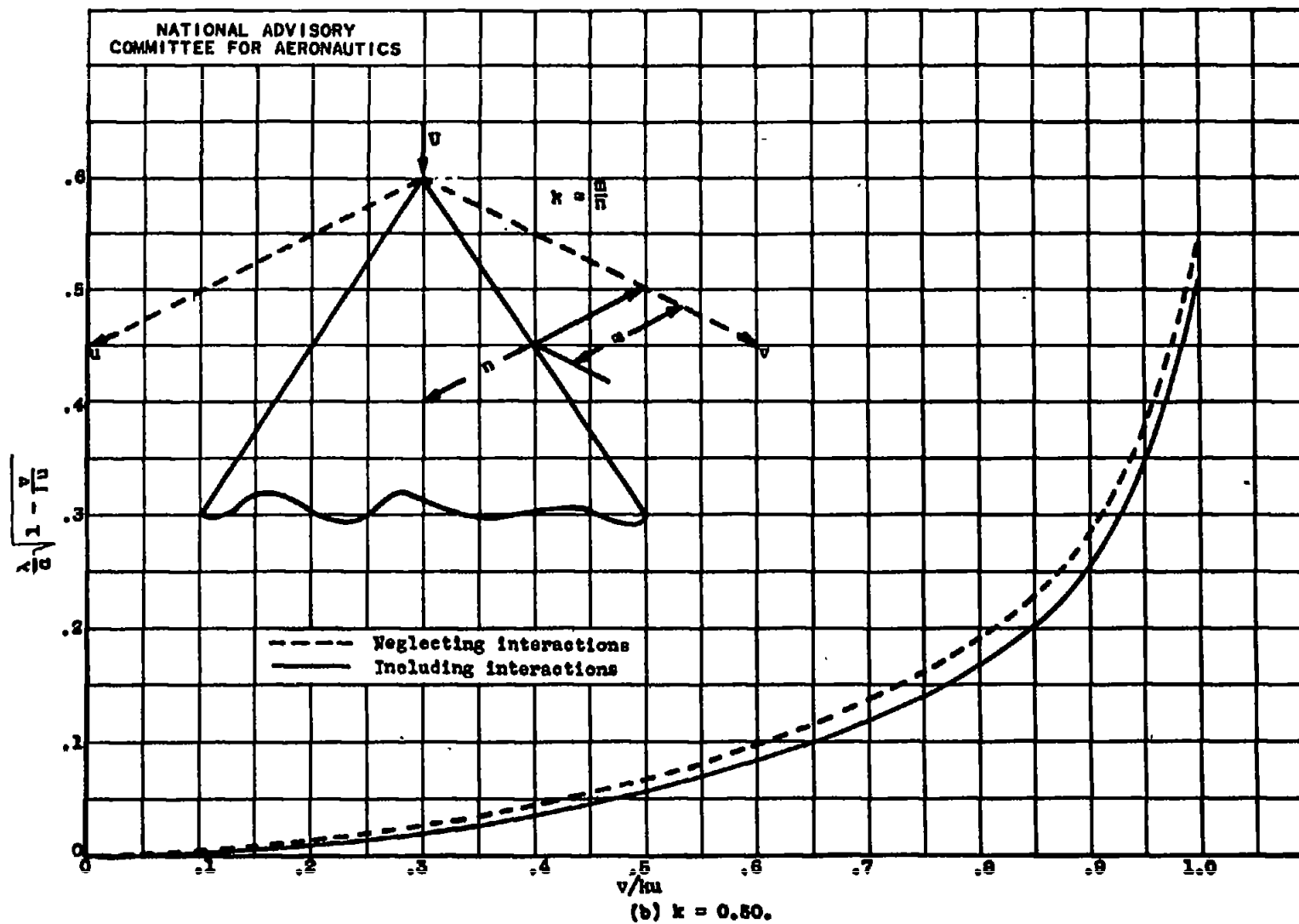


Figure 8. - Continued. Diaphragm slopes λ/a for symmetrical flat-plate delta wing.

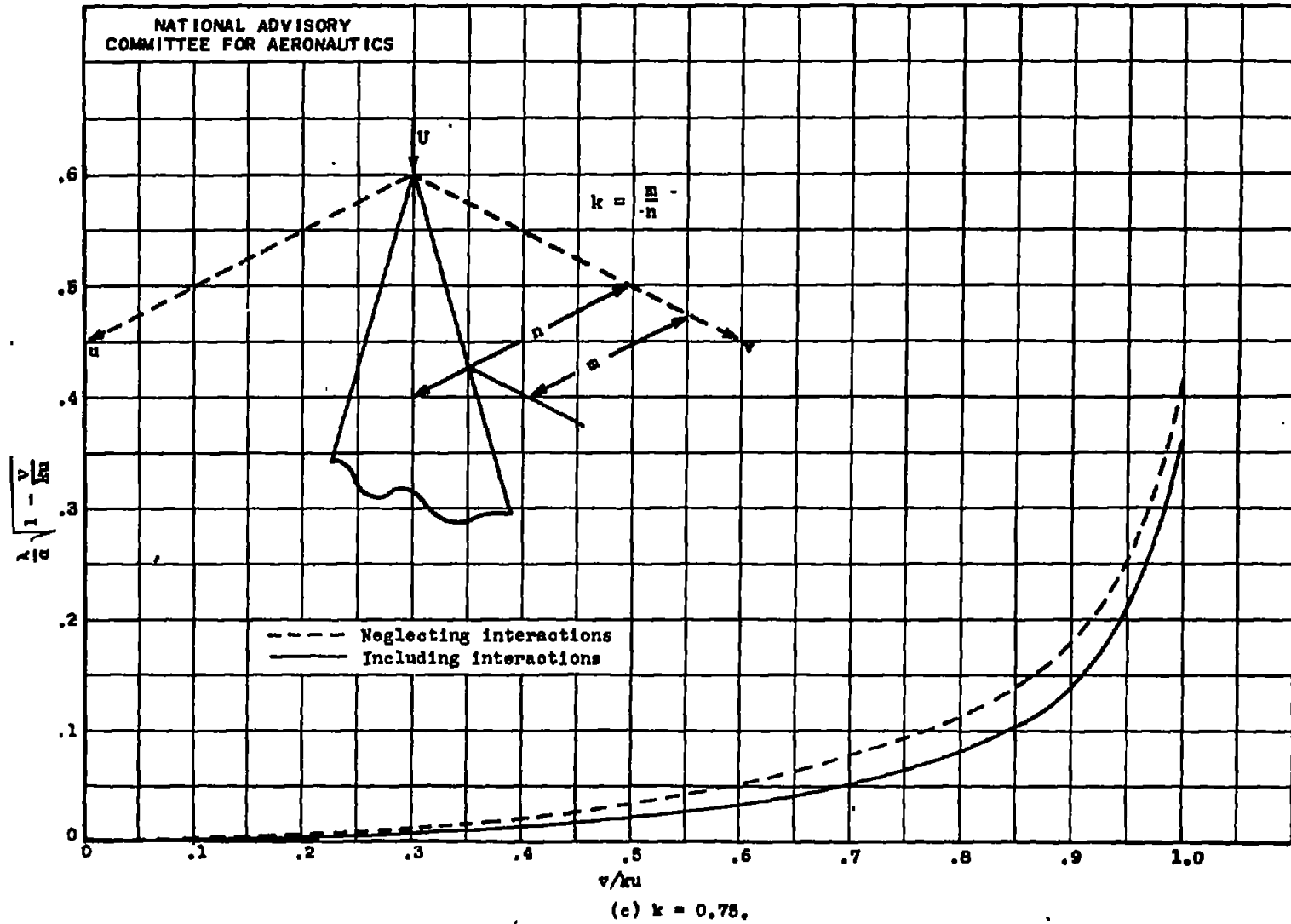


Figure 8. - Concluded. Diaphragm slopes λ/α for symmetrical flat-plate delta wing.

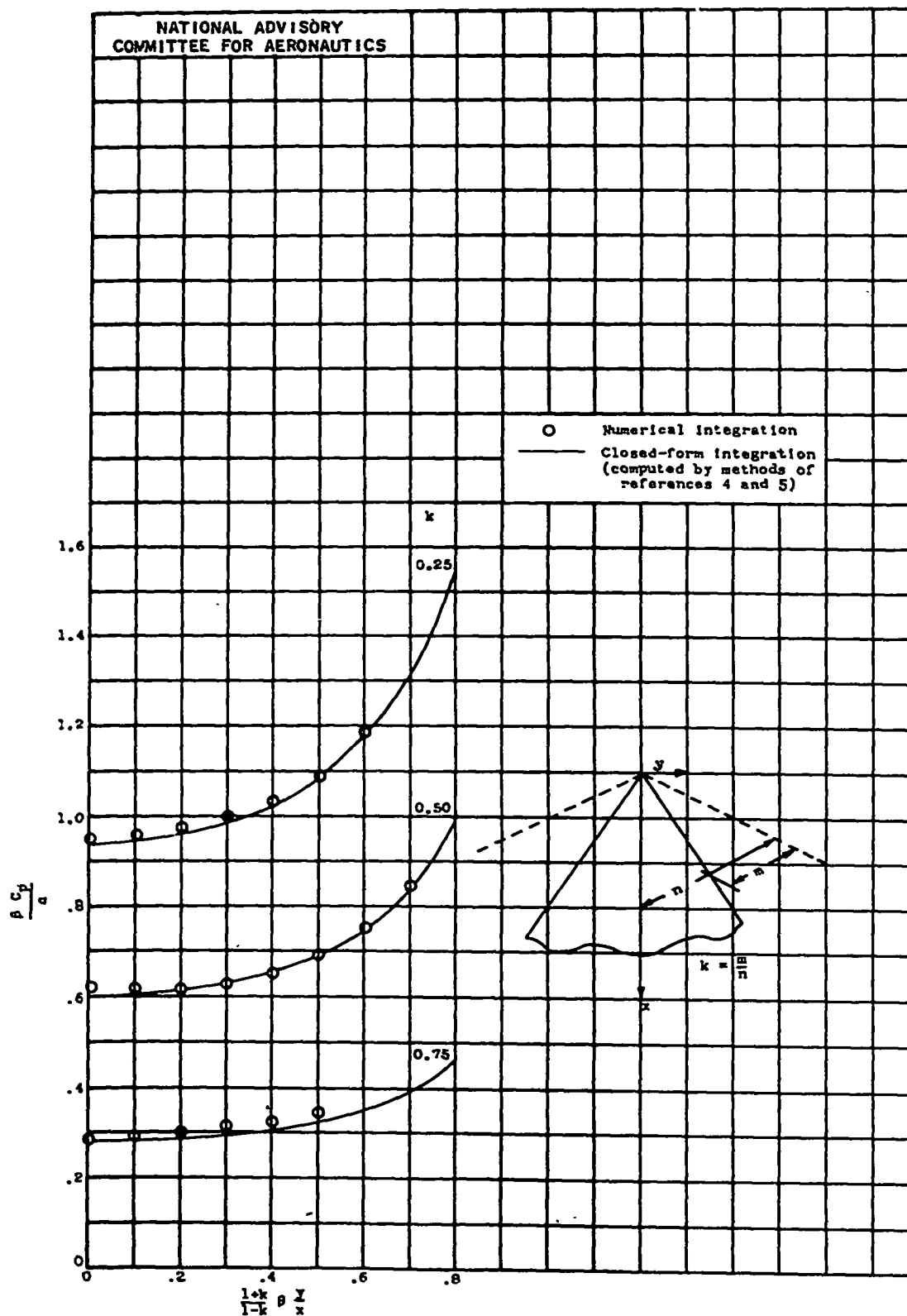


Figure 9. - Pressure coefficients of an unyawed flat-plate delta wing.